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On a two-phase free boundary problem

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Abstract
A two-phase free boundary problem associated with the Burgers equation is considered. The problem is reduced to a system of nonlinear integral equations which is analysed and shown to have a unique solution. The system admits a two-component shock solution which travels with the same velocity as that of the free boundary. The stability analysis of such a solution shows the existence of stability and instability regions according to different values of the parameters characterizing the system.

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1. Introduction

Free boundary problems (FBP) are relevant both in mathematics and in physics. From the mathematical point of view, the underlying complication of FBP involves not only solving the given partial differential equations, but also finding the unknown motion of the free boundary. Physically speaking, FBP arise in numerous contexts, e.g. surface dynamics in water waves, the internal evolution of the boundary between immiscible liquids, the motion of the free boundary between two states (e.g., Stefan problems), etc. For the state of the field of FBP up to 1984 we refer to [1–4]; more recent developments are reported in [5].

The inherent difficulty in most of the mentioned FBP is that they require one to solve a nonlinear system. In some cases it is possible to prove existence theorems (at least for short time) but, usually, special explicit solutions cannot be obtained. More recently, in [6] a one-phase Stefan problem for the Burgers equation was considered and an exact travelling wave solution was obtained. In [6] the existence of the one-phase solution is proved for short intervals of time. On the other hand, two-phase FBP are more complicated than their one-phase counterparts and the theory is more elaborate.

In this paper we concentrate on a particular nonlinear model which is well known in mathematics and physics: the Burgers equation (cf equations (1*a*) and (*b*)). Such an equation arises in weakly nonlinear gas dynamics [7]; it admits a linearization into the heat equation [8] and therefore is amongst the class of 'integrable' [9] equations. The free boundary problem that we study below corresponds to a one-dimensional, non-stationary flow of two weakly nonlinear compressible fluids. We assume the two fluids to be immiscible, with different velocity fields (excess of flow velocity over a sonic velocity), and different viscosity, connected by continuity of velocity and a suitable energy balance condition. Because the Burgers equation can be linearized into the heat equation, we have a natural correspondence with the well-known Stefan problem of the heat equation. For this reason we call this free boundary problem a Burgers–Stefan (BS) problem.

There are various significant differences between the systems (linear heat equation versus Burgers equation) which we note and which require special attention. Most significantly, the BS problem has a nonlinear convective term, arising from the governing fluidodynamic equations, which does not appear in the classical Stefan problems.

Notably, an exact solution to the BS problem can be determined; as such, this is unusual in the study of FBP where very few exact solutions are known. Moreover, we show how one can determine the stability of this solution. More concretely, in this paper we formulate and analyse a two-phase BS problem characterized by the following system of Burgers equations:

$$u_{1t} = \left(\delta u_{1x} - u_1^2\right)_x \qquad u_1 = u_1(x, t) \tag{1a}$$

defined over the domain $-\infty < x < s(t)$, $t > 0$ and

$$u_{2t} = \left(\delta_2 u_{2x} - u_2^2\right)_x \qquad u_2 = u_2(x, t) \tag{1b}$$

over the domain $s(t) < x < +\infty$, $t > 0$, where

$$s(0) = b > 0. \tag{1c}$$

Equations (1*a*) and (1*b*) have initial data given by

$$u_1(x, 0) = \psi_1(x) > 0 \qquad -\infty < x < b \qquad \psi_1(b) = 0. \tag{2a}$$

$$u_2(x, 0) = \psi_2(x) < 0 \qquad b < x < +\infty \qquad \psi_2(b) = 0 \tag{2b}$$

and the following set of boundary conditions:

$$u_1(-\infty, t) = \alpha_1 > 0 \qquad u_2(+\infty, t) = \beta_2 < 0 \tag{3a}$$

$$u_1(s(t), t) = u_2(s(t), t) = k. \tag{3b}$$

Equation (3*b*) in conjunction with a condition on the energy flux (see (3*c*)) is sufficient to determine the motion of the free boundary *s*(*t*). In principle, *k* in (3*b*) could be a given function of time. But, as in the linearized problem, here we will only discuss the fundamental case *k* being constant.

In the above relations, δ_j ($j = 1, 2$) are positive constants related to the viscosity of the two phases; α_1, *b* and *k* are positive constants (α_1, $>k$), β_2 is a negative constant and the unknown function *s*(*t*) describes the motion of the free boundary. The function *s*(*t*) has to be determined together with u_j (*x*, *t*) ($j = 1, 2$). Besides (3*b*), the system is characterized by an additional condition at the free boundary *s*(*t*), arising from energy consideration. The energy balance across the free boundary can be written as

$$\delta u_{2x}(s(t), t) - \tfrac{1}{2}u_2^2(s(t), t) - \delta u_{1x}(s(t), t) + \tfrac{1}{2}u_1^2(s(t), t) = (p_1 - p_2)\dot{s}(t) \tag{3c}$$

where p_1 and p_2 are the (constant) values of the pressure on the two sides of the boundary. Due to (3*b*), the kinetic energy terms on the right-hand side of (3*c*) cancel out and we can rewrite the above relation as

$$-\lambda_1 u_{1x}(s(t), t) + \lambda_2 u_{2x}(s(t), t) = \dot{s}(t) \tag{3d}$$

where $\lambda_j = (\delta_j/p)$, $p = p_1 - p_2$.

It is now worth noting that the Galilean transformations $x - kt \to x$, $u_j - k \to u_j$ $(j = 1, 2)$ leave equations (1*a*) and (1*b*) invariant while implying a trivial boundary datum in (3*b*). In the following we can then set $k = 0$ in (3*b*) without loss of generality.

Our analysis is based on the method developed in [6] for the solution of the one-phase Burgers–Stefan problem.

In the next section we reduce the two-phase BS problem to a system of coupled nonlinear integral equations and then prove the existence and uniqueness of its solution for small intervals of time. In section 3, we show that there is an explicit shock wave type solutions depending on the two components (a two-component shock wave solution is an exact special solution) of the two-phase BS problem, whose free boundary moves at the same velocity as the shock. In the final section we address the issue of stability of such a solution: we show that there exist regions of stability and instability of the shock wave according to different values of the parameters which characterize the two components of the solution.

2. Solution of the two-phase BS problem

We begin our analysis by introducing the generalized Hopf–Cole transformation [6]

$$u_j(x,t) = -v_j(x,t) \Bigg/ \left[C_j(t) + \frac{1}{\delta_j} \int_{s(t)}^{x} \mathrm{d}x'\, v_j(x',t) \right] \qquad j = 1, 2 \qquad (4a)$$

$$v_j(x,t) = C_j(t) u_j(x,t) \exp\left[-\frac{1}{\delta_j} \int_{s(t)}^{x} \mathrm{d}x'\, u_j(x',t) \right]. \qquad (4b)$$

with

$$C_j(0) = 1. \qquad (4c)$$

This transformation, with conditions (2) and (3), implies the relations

$$v_j(x,0) = \varphi_j(x) = \psi_j(x) \exp\left[-\frac{1}{\delta_j} \int_{b}^{x} \mathrm{d}x'\, \psi_j(x') \right] \qquad j = 1, 2 \qquad (5a)$$

with $\varphi_1(x) > 0,\ \varphi_2(x) < 0$ and $\varphi_j(b) = 0$,

$$v_j(s(t),t) = C_j(t) u_j(s(t),t) = 0 \qquad (5b)$$

$$-\lambda_1 \frac{v_{1x}(s(t),t)}{C_1(t)} + \lambda_2 \frac{v_{2x}(s(t),t)}{C_2(t)} = \dot{s}(t). \qquad (5c)$$

The transformation (4*a*)–(4*c*) maps the system (1*a*)–(1*c*) into the following system for the linear heat equation,

$$v_{jt} = \delta_j v_{j_{xx}} \qquad j = 1, 2 \qquad (6)$$

with the compatibility conditions

$$\dot{C}_j(t) = -v_{j_x}(s(t),t) \qquad j = 1, 2. \qquad (5d)$$

Thus the two-phase BS problem (1)–(3) is reduced to the two-phase Stefan problem for the heat equation (6) with the initial data (5*a*), characterized by the boundary conditions at the free boundary (5*b*)–(5*d*). We say that $\{v_j(x,t)\ (j = 1, 2), s(t)\}$ form a solution of the above Stefan problem for all $t < \sigma, 0 < \sigma < \infty$, when: (a) $v_j(x,t)$ is a solution of (6) satisfying (5*a*)–(5*d*), they exist and are continuous together with their derivatives; (b) $s(t)$ is a continuously differentiable function for $0 \leqslant t < \sigma$.

In order to prove the existence and uniqueness of the solution for $t < \sigma$, we assume that the initial data $\psi_j(x)$ $(j = 1, 2)$ given in (2*a*) and (2*b*) are continuous together with their derivatives; moreover, they are bounded:

$$|\psi_1(x)| < \alpha_1 \qquad |\psi_2(x)| < |\beta_2|$$

with α_1 and β_2 given by (3*a*).

Next we observe that the unknown functions $C_j(t)$ entering the transformation (4*a*)–(4*c*) satisfy the relation

$$\lambda_1 \ln C_1(t) - \lambda_2 \ln C_2(t) = s(t) - b \tag{7a}$$

which is obtained from (5*c*) together with the compatibility condition (5*d*). Integrating (5*d*) and substituting in (7*a*), we obtain

$$\lambda_1 \ln\left[1 - \int_0^t v_{1x}(s(t'), t')\,\mathrm{d}t'\right] - \lambda_2 \ln\left[1 - \int_0^t v_{2x}(s(t'), t')\,\mathrm{d}t'\right] = s(t) - b. \tag{7b}$$

We now turn our attention to the solution of (6). To this end we introduce the fundamental kernel of the heat equation

$$K_j(x-\xi, t-t') = \frac{1}{2\sqrt{\pi\delta_j}}\frac{1}{\sqrt{t-t'}}\exp\left[-\frac{(x-\xi)^2}{4\delta_j(t-t')}\right] \qquad j = 1, 2 \tag{8}$$

and integrate Green's identity for the heat equation

$$\frac{\partial}{\partial\xi}\left(K_j\frac{\partial v_j}{\partial\xi} - v_j\frac{\partial K_j}{\partial\xi}\right) - \frac{\partial}{\partial t}(K_j v_j) = 0 \qquad j = 1, 2 \tag{9}$$

over the domain $-\infty < \xi < s(t')$ in the case $j = 1$ $(s(t') < \xi < +\infty$ for $j = 2)$, $0 < \varepsilon < t' < t - \varepsilon$ and let $\varepsilon \to 0$. Using $v(s(t'), t') = 0$ and $K_j(x - \xi, 0) = \delta(x - \xi)$ we obtain

$$v_1(x,t) = \int_{-\infty}^{b} K_1(x-\xi, t)\varphi_1(x)\,\mathrm{d}\xi + \int_0^t K_1(x - s(t'), t-t')v_{1x}(s(t'), t')\,\mathrm{d}t' \tag{10a}$$

$$v_2(x,t) = \int_{b}^{\infty} K_2(x-\xi, t)\varphi_2(x)\,\mathrm{d}\xi - \int_0^t K_2(x - s(t'), t-t')v_{2x}(s(t'), t')\,\mathrm{d}t' \tag{10b}$$

with $s(t)$ given by (7*b*).

On the right-hand sides of (10*a*) and (10*b*) $v_{jx}(s(t), t)$ $(j = 1, 2)$ is unknown; it is then convenient to take the *x*-derivative of both sides in (10*a*) and (10*b*) and take its limit as $x \to s(t)^-$ $(x \to s(t)^+)$.

We then put $z_j(t) = v_{xj}(s(t), t)$ $(j = 1, 2)$ and finally obtain (cf [5])

$$z_1(t) = \int_{-\infty}^{b} K_1(s(t)-\xi, t)\varphi_1'(\xi)\,\mathrm{d}\xi + 2\int_0^t K_{1x}(s(t) - s(t'), t-t')z_1(t')\,\mathrm{d}t' \tag{11a}$$

$$z_2(t) = \int_{b}^{\infty} K_2(s(t)-\xi, t)\varphi_2'(\xi)\,\mathrm{d}\xi + 2\int_0^t K_{2x}(s(t) - s(t'), t-t')z_2(t')\,\mathrm{d}t' \tag{11b}$$

with

$$\lambda_1 \ln\left[1 - \int_0^t z_1(t')\,\mathrm{d}t'\right] - \lambda_2 \ln\left[1 - \int_0^t z_2(t')\,\mathrm{d}t'\right] = s(t) - b. \tag{11c}$$

Thus the solution of the Stefan problem (6), (5*a*)–(5*d*) has been reduced to the solution of the system of coupled nonlinear integral equations (11*a*)–(11*c*).

Once the existence and uniqueness of the functions $z_j(t)$ $(j = 1, 2)$ is proved for $0 \leqslant t < \sigma$, the existence and uniqueness of $v_j(x, t)$ $(j = 1, 2)$ then follows, via (10*a*) and (10b). The solution of the two-phase Burgers–Stefan problem (1)–(3) then exists and is unique (for $0 \leqslant t < \sigma$) due to (4a), with $C_j(t)$ obtained via (5b).

In order to analyse existence properties of $z_1(t)$ and $z_2(t)$ for $0 \leqslant t < \sigma$, we denote by S_M the closed sphere $\|z\| \leqslant M$ in the Banach space of functions $z(t)$ continuous for $0 \leqslant t < \sigma$, with the uniform norm $\|z\| = \text{l.u.b.}|z(t)|$. On the sphere S_M we define the mappings

$$w_j(t) = T_j z_j(t) \qquad j = 1, 2 \tag{12}$$

where $T_1 z_1$ and $T_2 z_2$ coincide with the right-hand sides of (11a) and (11b), respectively. We first prove that T_j ($j = 1, 2$) is a mapping of S_M into itself.

From (11c) we obtain (see (A1))

$$|s(t)| \leqslant b + 2M\lambda\sigma \tag{13}$$

with $\lambda = \lambda_1 + \lambda_2$ and $2M\sigma < 1$ (cf [7], p 161).

It then follows that

$$b \leqslant |s(t)| \leqslant b + 2M\lambda\sigma. \tag{14}$$

From (11c) we also obtain (see (A2c))

$$|s(t) - s(t')| \leqslant 2M\lambda|t - t'|. \tag{15}$$

We now consider the right-hand side of equation (12) in the case $j = 1$. It is shown in the appendix (cf (A3)–(A5)) that

$$\|w_1\| = \|T_1 z_1\| \leqslant 2A\,\mathrm{e}^{b\gamma_1} + \frac{2M^2\lambda}{\sqrt{\pi}\,\delta_1^{3/2}}\sqrt{\sigma} \tag{16}$$

where $A \equiv \left(\|\psi_1'\| + \alpha_1^2\right)$ (cf (A3)) and $\gamma_1 = \frac{\alpha_1}{\delta_1}$.

We now define M as $M = \max(M_1, M_2)$, M_j: $M_j = 1 + 2A\,\mathrm{e}^{b\gamma_j}$ ($j = 1, 2$) and take $\sigma < \min(\sigma_1, \sigma_2)$, σ_1: $2M\lambda\sigma_1 < 1$, σ_2: $3M^2\lambda\sqrt{\sigma} < \sqrt{\pi}(\delta_1)^{3/2}$. It then follows from (16) that

$$\|w_1\| = \|T_1 z_1\| \leqslant M. \tag{17a}$$

Along the same lines it is possible to show that

$$\|w_2\| = \|T_2 z_2\| \leqslant M. \tag{17b}$$

(17a) and (17b) imply that the mappings T_1 and T_2 are closed.

Next, we prove that T_j ($j = 1, 2$) is a contraction; i.e. given two solutions of (12) with $\|z_j - \bar{z}_j\| = d$, $d < 2M$, it follows that $\|T_1(z_j - \bar{z}_j)\| \leqslant \vartheta d$ with $0 < \vartheta < 1$.

To this end we denote by B_i appropriate positive constants and obtain from (11c) the following relevant bounds (cf (A6a) and (A6b)):

$$|s(t) - \bar{s}(t)| < B_1 dt \qquad (B_1 = 2\lambda) \tag{18a}$$

and

$$|\dot{s}(t) - \dot{\bar{s}}(t)| < B_2 d \qquad \left(B_2 = \lambda\,\mathrm{e}^{(1+\mathrm{e}^{2d\sigma})}\right). \tag{18b}$$

We now consider the case $j = 1$. From equations (12) and (11a) we can write

$$w_1 - \bar{w}_1 = H_1 + H_2 \tag{19a}$$

$$H_1 = \frac{1}{\sqrt{\pi t \delta_1}} \int_{-\infty}^{b} \varphi_1'(\xi) \left[\exp\left(-\frac{(s(t) - \xi)^2}{4\delta_1 t}\right) - \exp\left(-\frac{(\bar{s}(t) - \xi^2)}{4\delta_1 t}\right)\right] \mathrm{d}\xi \tag{19b}$$

$$H_2 = -\int_0^t \mathrm{d}t'\, z_1(t') \frac{(s(t)-s(t'))}{\delta_1(t-t')} K_1(s(t)-s(t'),t-t')$$
$$+\int_0^t \mathrm{d}t'\, \bar{z}_1(t') \frac{(\bar{s}(t)-\bar{s}(t'))}{\delta_1(t-t')} K_1(s(t)-s(t'),t-t') \qquad (19c)$$

with $K_1(x-\xi,t-t')$ given by (8).

We derive in appendix A (cf (A7)) the following bound on H_1:

$$|H_1| < \frac{A\,\mathrm{e}^{b\gamma_1}}{\sqrt{\pi}\,\delta_1} B_1 d\sqrt{\sigma} \equiv B_3 d\sqrt{\sigma}. \qquad (20)$$

Next, the estimation of H_2 is obtained by writing

$$|H_2| \leqslant |V_1| + |V_2| + |V_3| \qquad (21a)$$

$$V_1 = -\int_0^t \mathrm{d}t'\, (z(t')-\bar{z}(t')) \frac{(s(t)-s(t'))}{\delta_1(t-t')} K_1(s(t)-s(t'),t-t') \qquad (21b)$$

$$V_2 = -\int_0^t \mathrm{d}t'\, \bar{z}(t') \left[\frac{(s(t)-s(t'))}{\delta_1(t-t')} - \frac{(\bar{s}(t)-\bar{s}(t'))}{\delta_1(t-t')}\right] K_1(s(t)-s(t'),t-t') \qquad (21c)$$

$$V_3 = -\int_0^t \mathrm{d}t' \bar{z}(t') \left[\frac{(\bar{s}(t)-\bar{s}(t'))}{\delta_1(t-t')}\right] K_1(s(t)-s(t'),t-t')$$
$$\times \left[1 - \exp\left\{\frac{(\bar{s}(t)-\bar{s}(t'))^2 - (s(t)-s(t'))^2}{4\delta_1(t-t')}\right\}\right]. \qquad (21d)$$

In appendix A the following bounds are obtained (cf (A8)–(A10)):

$$|V_1| < \frac{B_1 M d\sqrt{\sigma}}{\sqrt{\pi}\delta_1^{3/2}} \equiv B_4 d\sqrt{\sigma} \qquad (22a)$$

$$|V_2| < \frac{B_2 M d\sqrt{\sigma}}{\sqrt{\pi}\delta_1^{3/2}} \equiv B_5 d\sqrt{\sigma} \qquad (22b)$$

$$|V_3| < B_8 d\sqrt{\sigma} \qquad (22c)$$

where B_8 is defined in (A10).

Combining (22a)–(22c) we have from (21a)

$$|H_2| < (B_4 + B_5 + B_8) d\sqrt{\sigma} \equiv B_9 d\sqrt{\sigma}. \qquad (23)$$

From (19a), (20) and (23) we finally get

$$\frac{\|w_1 - \bar{w}_1\|}{d} < (B_3 + B_9)\sqrt{\sigma} \equiv B_{10}\sqrt{\sigma}. \qquad (24)$$

If we choose σ to satisfy $\sigma < \min(\sigma_1, \sigma_2, \sigma_3)$ with

$$B_9\sqrt{\sigma_3} < 1 \qquad (25)$$

it then follows that T_1 is a contraction operator on S_M. Following the same lines it can be proved that also T_2 is a contraction operator in S_M. We therefore conclude that $z_1(t) = T_1 z_1(t)$ and $z_2(t) = T_2 z_2(t)$ exist and are unique fixed points of T_1 and T_2 in S_M, for $0 \leqslant t < \sigma$.

We have thus proved that the solution of the system of nonlinear integral equations (11a)–(11c) exists and is unique for a small interval of time.

In the next section we turn our attention to an explicit, particular solution of the two-phase BS problem.

3. Two-component shock wave

We write the usual shock solution of (1*a*) compatible with (2*a*) and (3*a*); it reads

$$u_1(x,t) = \alpha_1 + \frac{(\alpha_2 - \alpha_1)}{\left[1 + \exp \frac{1}{\delta_1}(\alpha_2 - \alpha_1)(x - V_1 t - x_0')\right]} \tag{26a}$$

with

$$V_1 = \alpha_1 + \alpha_2 \qquad \alpha_2 < 0 < \alpha_1. \tag{26b}$$

The corresponding solution of (1*b*) satisfying (2*b*) and (3*a*) reads

$$u_2(x,t) = \beta_1 + \frac{(\beta_2 - \beta_1)}{\left[1 + \exp \frac{1}{\delta_2}(\beta_2 - \beta_1)(x - V_2 t - x_0'')\right]} \tag{27a}$$

with

$$V_2 = \beta_1 + \beta_2 \qquad \beta_2 < 0 < \beta_1. \tag{27b}$$

In the above relations α_2 and β_1 are constants to be determined.

We use (26*a*) ((27*a*)) on the interval $-\infty < x < s(t)$ $(s(t) < x < +\infty)$ and require

$$u_1(x,t) = 0 \qquad x > s(t) \qquad (u_2(x,t) = 0, x < s(t)).$$

We now impose on $u_1(x,t)$ and $u_2(x,t)$ the condition at the free boundary (3*b*); we get

$$s(t) - V_1 t = x_0' + \frac{\delta_1 s_0'}{(\alpha_1 - \alpha_2)} \tag{28a}$$

and

$$s(t) - V_2 t = x_0'' + \frac{\delta_2 s_0''}{(\beta_1 - \beta_2)} \tag{28b}$$

which imply that the shock solutions (26*a*) and (27*a*) are both moving with the same velocity as the free boundary

$$\dot{s}(t) = V_1 = V_2 \equiv V. \tag{28c}$$

Next, the boundary condition (3*c*) implies

$$-\frac{\lambda_1}{\delta_1}\alpha_1\alpha_2 + \frac{\lambda_2}{\delta_2}\beta_1\beta_2 = V. \tag{29}$$

Equations (28*c*) and (29), when (26*b*) and (27*b*) are also used, fix the value of the constants α_2, α_1 and V:

$$\alpha_2 = \frac{\alpha_1\left(1 - \frac{\lambda_2}{\delta_2}\beta_2\right) + \frac{\lambda_2}{\delta_2}\beta_2^2}{\left(\frac{\lambda_2}{\delta_2}\beta_2 - \frac{\lambda_1}{\delta_1}\alpha_1 - 1\right)} \tag{30a}$$

$$\beta_1 = \frac{\beta_2\left(1 + \frac{\lambda_1}{\delta_1}\alpha_1\right) - \frac{\lambda_1}{\delta_1}\alpha_1^2}{\left(\frac{\lambda_2}{\delta_2}\beta_2 - \frac{\lambda_1}{\delta_1}\alpha_1 - 1\right)} \tag{30b}$$

$$V = \frac{\frac{\lambda_2}{\delta_2}\beta_2^2 - \frac{\lambda_1}{\delta_1}\alpha_1^2}{\left(\frac{\lambda_2}{\delta_2}\beta_2 - \frac{\lambda_1}{\delta_1}\alpha_1 - 1\right)}. \tag{30c}$$

Due to (3a), it is immediate to verify that (30a) and (30b) imply $\alpha_2 < 0$ and $\beta_1 > 0$ respectively; also, it follows from (30c) that $V > 0$ provided

$$\frac{\lambda_2}{\delta_2}\beta_2^2 < \frac{\lambda_1}{\delta_1}\alpha_1^2.$$

Finally, via (28a) and (28b), the position of the boundary is fixed as

$$\exp\left(-\frac{s_0'}{\delta_1}\right) = \frac{|\alpha_2|}{\alpha_1} \tag{31}$$

and

$$\exp\left(-\frac{s_0''}{\delta_2}\right) = \frac{|\beta_2|}{\beta_1}. \tag{32}$$

4. Stability analysis and results

In order to study the stability of the particular solution $\{u_j(x,t)\ (j=1,2), s(t)\}$ obtained in the previous section, we consider small perturbations affecting both the shocks and the motion of the free boundary. We set

$$u_j = \hat{u}_j + u_j' \qquad j = 1, 2 \tag{33}$$

$$s(t) = \hat{s}(t) + s'(t) \tag{34}$$

where $\hat{u}_j\ (j = 1, 2)$ is the shock solution satisfying $\hat{u}_j(\hat{s}(t), t) = 0$ and u_j', s' are small perturbations.

By linearizing (1a) and (1b) around $\hat{u}_j$, we get

$$\vartheta_{jt} = \delta_j \vartheta_{jxx} - 2\hat{u}_j \vartheta_{jx} \qquad j = 1, 2 \tag{35}$$

where the position $u_j' = \vartheta_{jx}$ has been made.

The boundary conditions (3a) and (3b), together with (33) and (34), give the conditions for $\vartheta_j(x, t)$ at the free boundary:

$$\frac{\delta_1}{\alpha_1\alpha_2}\vartheta_{1x}(\hat{s}(t), t) = \frac{\delta_2}{\beta_1\beta_2}\vartheta_{2x}(\hat{s}(t), t) \tag{36a}$$

and

$$\frac{\delta_1}{\alpha_1\alpha_2}\frac{\partial}{\partial t}\vartheta_{1x}|_{x=\hat{s}(t)} = \left[(\lambda_2\vartheta_{2xx} - \lambda_1\vartheta_{1xx}) + V\left(\frac{\lambda_2}{\delta_2}\vartheta_{2x} - \frac{\lambda_1}{\delta_1}\vartheta_{1x}\right)\right]_{x=\hat{s}(t)}. \tag{36b}$$

The change of variables

$$\vartheta_j(x, t) = \vartheta_j(X, t) \qquad X = x - Vt \qquad j = 1, 2 \tag{37}$$

maps (35) into

$$\vartheta_{jt} = \delta_j \vartheta_{jXX} - (2\hat{u}_j - V)\vartheta_{jX} \qquad j = 1, 2 \tag{38}$$

and (36a) and (37b) into

$$\frac{\delta_1}{\alpha_1\alpha_2}\vartheta_{1X}(0, t) = \frac{\delta_2}{\beta_1\beta_2}\vartheta_{2X}(0, t) \tag{39a}$$

and

$$\frac{\delta_1}{\alpha_1\alpha_2}(\vartheta_{1tX} - V\vartheta_{1XX})|_{x=0} = \left[(\lambda_2\vartheta_{2XX} - \lambda_1\vartheta_{1XX}) + V\left(\frac{\lambda_2}{\delta_2}\vartheta_{2X} - \frac{\lambda_1}{\delta_1}\vartheta_{1X}\right)\right]_{X=0} \tag{39b}$$

respectively.

We now solve (38) with the initial condition

$$\vartheta_j(X,0) = f_j(X) \qquad j = 1,2 \tag{39c}$$

and the asymptotically vanishing condition $\vartheta_1 \to 0$ ($\vartheta_2 \to 0$) as $X \to -\infty$ ($X \to +\infty$).

In terms of the Laplace transform

$$\hat{\vartheta}_j(X,q) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-qt}\vartheta_j(X,t) \qquad j = 1,2 \tag{40}$$

from (38) and (39*a*)–(39*c*) we get the solutions

$$\hat{\vartheta}_1(X,q) = \exp(-P_1(X))\left[c_1\,\mathrm{e}^{k_1X} + \int_0^X \frac{\mathrm{e}^{k_1(X-\xi)}}{2k_1}F_1(\xi)\,\mathrm{d}\xi - \int_{-\infty}^X \frac{\mathrm{e}^{-k_1(X-\xi)}}{2k_1}F_1(\xi)\,\mathrm{d}\xi\right] \tag{41a}$$

and

$$\hat{\vartheta}_2(X,q) = \exp(-P_2(X))\left[c_2\,\mathrm{e}^{-k_2X} - \int_0^X \frac{\mathrm{e}^{-k_2(X-\xi)}}{2k_2}F_2(\xi)\,\mathrm{d}\xi - \int_X^\infty \frac{\mathrm{e}^{k_2(X-\xi)}}{2k_2}F_2(\xi)\,\mathrm{d}\xi\right] \tag{41b}$$

with

$$P_j(X) = \int_0^X \left(\frac{V}{2} - \hat{u}_j(X')\right)\mathrm{d}X' \qquad j = 1,2 \tag{41c}$$

$$k_1 = \left(\frac{1}{4\delta_1^2}(\alpha_1-\alpha_2)^2 + \frac{q}{\delta_1}\right)^{1/2} \tag{41d}$$

$$k_2 = \left(\frac{1}{4\delta_2^2}(\beta_1-\beta_2)^2 + \frac{q}{\delta_2}\right)^{1/2} \tag{41e}$$

and

$$F_j(X) = -f_j(X)\exp(P_j(X)) \qquad j = 1,2. \tag{41f}$$

c_1 and c_2 in (41*a*) and (41*b*) have to be determined via the boundary conditions (39*a*) and (39*b*).

The small perturbation $u'_j(X,t)$, $j = 1,2$, is finally obtained by inverting (41*a*) and (41*b*) and taking the *x*-derivative. When the large time behaviour of $u'_j(X,t)$ is considered, we observe that all the contributions coming from the integral terms of (41*a*) and (41*b*) are asymptotically vanishing as $t \to \infty$, since the branch points q_j of the solution are real and negative. We therefore conclude that the only possible source of asymptotically non-vanishing contributions to $u'_j(X,t)$ is determined by the positive singularities of c_1 and c_2.

When the boundary conditions (39*a*) and (39*b*) are imposed on (41*a*) and (41*b*), one obtains for c_1 and c_2 a system of the form

$$A_{11}c_1 + A_{12}c_2 = 0 \tag{42a}$$

$$A_{21}c_1 + A_{22}c_2 = H \tag{42b}$$

where A_{ij} and H ($i = 1,2$ and $j = 1,2$) depend on the variable q, and on the parameters k_j, λ_j, δ_j, α_1 and β_2. Their explicit form is given in appendix B.

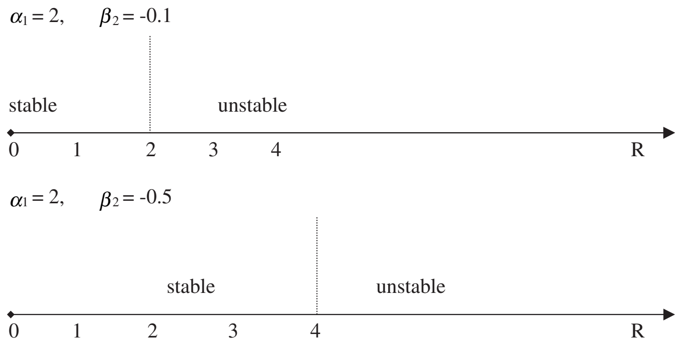

Figure 1. Regions of stability and instability of the shock solution are sketched for different values of the parameters (α_1, β_2) versus the ratio $R = \frac{\lambda_2}{\delta_2}/\frac{\lambda_1}{\delta_1}$.

The determinant Δ of the system (42*a*)–(42*b*) $(\Delta = A_{11}A_{22} - A_{21}A_{12})$ can be evaluated as a function of q for different values of the parameters λ_j, δ_j, α_1, β_2.

The zeros of Δ (for q positive) determine the instability of the shock solution $\hat{u}_j$ $(j = 1, 2)$ with respect to the small perturbation u'_j.

The results of numerical computations indicate that the behaviour of Δ as a function of q is strongly influenced by the values of the parameters λ_j, δ_j, α_1 and β_2. On the other hand, the results of [6] show that the shock wave is a stable solution of the one-phase Burgers–Stefan problem. Such a solution can be recovered from (30*a*)–(30*c*) in the limit $\frac{\lambda_2}{\delta_2}|\beta_2| \ll \frac{\lambda_1}{\delta_1}\alpha_1$ with $|\beta_2|$ small.

The result of the numerical computations is shown in figure 1, where regions of stability and instability of the shock wave are indicated for two sets of values (α_1, β_2) as a function of the ratio $R = \frac{\lambda_2}{\delta_2}\Big/\frac{\lambda_1}{\delta_1}$.

For the first set of values we choose $\alpha_1 = 2, \beta_2 = -0.1, \frac{\lambda_2}{\lambda_1} = 0.1$; for the second set we fix instead $\alpha_1 = 2, \beta_2 = -0.5, \frac{\lambda_2}{\lambda_1} = 4$. We see from figure 1 that the shock solution for the first set of parameters is stable in the region $0 \leqslant R \leqslant R_0,\ R_0 \cong 2$, and unstable for $R > R_0$; for the second set the stability region corresponds instead to $0 \leqslant R \leqslant R_1,\ \ R_1 \cong 4$, and instability is determined by $R \geqslant R_1$. In both cases the velocity V given by (30*c*) is positive as long as there holds the condition $\frac{\lambda_2}{\delta_2}\beta_2^2 < \frac{\lambda_1}{\delta_1}\beta_1^2$.

V will eventually become negative in the instability regions when the ratio R is increased sufficiently so that the previous inequality is not satisfied anymore. Thus the velocity need not be of any fixed sign in order to have stability.

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Appendix A

In the following we derive some relevant inequalities of section 2. We start with the estimate of $s(t)$; by introducing the notation $\xi_j(t) = \int_0^t z_j(t')\,\mathrm{d}t'$, $j = 1, 2$, we obtain from (11*c*)

$$|s(t)| \leqslant b + \sum_{j=1}^{2} \lambda_j |\ln(1-\xi_j(t))| \leqslant b + 2\sum_{j=1}^{2} \lambda_j |\xi_j(t)|$$
$$\leqslant b + 2\lambda_1 \int_0^{\sigma} |z_1(t')|\,\mathrm{d}t' + 2\lambda_2 \int_0^{\sigma} |z_2(t')|\,\mathrm{d}t' \leqslant b + 2\lambda M\sigma \qquad \text{(A1)}$$

where $\lambda = \lambda_1 + \lambda_2$ and $2M\sigma < 1$ (cf [7], fg |6|).

In order to prove (15) we use again (11*c*) and write

$$s(t) - s(t') = \lambda_1[\ln(1-\xi_1(t)) - \ln(1-\xi_1(t'))] - \lambda_2[\ln(1-\xi_2(t)) - \ln(1-\xi_2(t'))] \qquad \text{(A2}a)$$

where t is $|\xi_j(t)| < 1/2$, $j = 1, 2$.

We then expand in Taylor series the right-hand side of (A2*a*) and obtain

$$|\ln(1-\xi_j(t)) - \ln(1-\xi_j(t'))| \leqslant \left| \sum_{n=1}^{\infty} \frac{\left(\xi_j^n(t') - \xi_j^n(t)\right)}{n} \right| \leqslant |\xi_j(t') - \xi_j(t)| \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$
$$\leqslant 2|\xi_j(t') - \xi_j(t')| \qquad j = 1, 2. \qquad \text{(A2}b)$$

Next we consider the norm $||W_1||$ and prove relation (16). From (5*a*) we observe that

$$|\varphi_1'(x)| \leqslant \left|\psi_1'(x) + \psi_1^2(x)\right| \exp\left(\frac{1}{\delta_1}\int_x^b \psi_1(x')\,\mathrm{d}x'\right)$$
$$\leqslant \left(\left\|\psi_1^1\right\| + \alpha_1^2\right) \exp\left((b-x)\,\frac{\alpha_1}{d_1}\right) \equiv A \exp\left((b-x)\,\frac{\alpha_1}{d_1}\right). \qquad \text{(A3)}$$

Moreover, we note that the two terms on the right-hand side of (11*a*) satisfy

$$2\left|\int_{-\infty}^{b} k_1\,(s\,(t)-\xi,t)\,\varphi_1'\,(\xi)\,\mathrm{d}\xi\right|$$
$$\leqslant \frac{A}{\sqrt{\pi\delta_1 t}} \exp\left(\frac{b\alpha_1}{\delta_1}\right) \int_{-\infty}^{b} \exp\left(\frac{(s\,(t)-\xi)^2}{4\delta_1 t}\right) \exp\left(\frac{-\alpha_1\xi}{\delta_1}\right) \mathrm{d}\xi$$
$$\leqslant 2A\,\mathrm{e}^{b\gamma_1} \qquad (\gamma_1 = \alpha_1/\delta_1) \qquad \text{(A4}a)$$

$$\frac{1}{2\sqrt{\pi}}\frac{1}{\delta_1^{3/2}}\left|\int_0^t \frac{(s(t)-s(t'))}{(t-t')^{3/2}} \exp\left[-\frac{(s(t)-s(t'))^2}{4\delta_1(t-t')}\right] z_1(t')\,\mathrm{d}t'\right|$$
$$\leqslant \frac{M}{2\sqrt{\pi}}\frac{1}{\delta_1^{3/2}} \int_0^t \frac{|s(t)-s(t')|}{(t-t')^{3/2}} \leqslant \frac{M^2\lambda}{\sqrt{\pi}\delta_1^{3/2}}\sqrt{\sigma}. \qquad \text{(A4}b)$$

When the above relations are used, from (12) and (11*a*) we finally get

$$\|w_1\| \leqslant 2A\,\mathrm{e}^{b\gamma_1} + \frac{2M^2\lambda}{\sqrt{\pi}\delta_1^{3/2}}\sqrt{\sigma}. \qquad \text{(A5)}$$

In order to prove (18*a*) and (18*b*) we consider (11*c*) and write

$$s(t) - \bar{s}(t) = \lambda_1[\ln(1-\xi_1(t)) - \ln(1-\bar{\xi}_1(t))] - \lambda_2[\ln(1-\xi_2(t)) - \ln(1-\bar{\xi}_2(t))].$$

We now make use of the inequality

$$|\ln(1-\xi_j(t)) - \ln(1-\bar{\xi}_j(t))| \leqslant 2\left|\int_0^t (z_j(t') - \bar{z}_j(t'))\,\mathrm{d}t'\right| \qquad j = 1, 2$$

and obtain

$$|s(t) - \bar{s}(t)| \leqslant 2(\lambda_1 + \lambda_2)\,\mathrm{d}t < B_1\mathrm{d}\sigma \qquad B_1 \equiv 2\lambda.$$

Next we consider the difference

$$\dot{s}(t) - \dot{\bar{s}}(t) = \sum_{j=1,2} \lambda_j [\bar{z}_j(t) \exp(-\ln(1-\bar{\xi}_j(t))) - z_j(t)\exp(-\ln(1-\xi_j(t)))]$$
$$= \sum_{j=1,2} \lambda_j \exp(-\ln(1-\bar{\xi}_j(t))) \bigg[(\bar{z}_j(t) - z_j(t))$$
$$+ z_j(t) \left(1 - \exp\left(\ln \frac{(1-\bar{\xi}_j(t))}{(1-\xi_j(t))}\right)\right)\bigg] \tag{A6a}$$

and use the inequalities $\ln(1-\xi_j(t))| \leqslant 2|\xi_j(t)| \leqslant 1$ and $|1-\mathrm{e}^{-Q}| \leqslant |Q|\,\mathrm{e}^{|Q|}$ where is $Q = -\ln\left[\frac{(1-\bar{\xi}_j(t))}{(1-\xi_j(t))}\right]$.

We finally get

$$|\dot{s}(t) - \dot{\bar{s}}(t)| \leqslant e \sum_{j=1,2}^{2} \lambda_j [d + 2M|\xi_j(t) - \bar{\xi}_j(t)| \exp(2|\xi_j(t) - \bar{\xi}_j(t)|)]$$
$$\leqslant e\lambda d(1 + 2M\sigma\,\mathrm{e}^{2d\sigma}) \leqslant B_2 d \tag{A6b}$$

with $B_2 = \lambda e(1+\mathrm{e}^{2d\sigma})$ and $2M\sigma < 1$. Our final task is to derive the bounds on $|H_1|$ and $|H_2|$. We start with (19*b*) and write

$$|H_1| \leqslant \frac{A\,\mathrm{e}^{b\gamma_1}}{\sqrt{\pi t \delta_1}} \int_{b-\bar{\xi}_0}^{b-\xi_0} \exp\left(-\frac{y^2}{4\delta_1 t}\right) \mathrm{d}y$$

where $\xi_0 = s(t) - 2t\gamma_1$, $\bar{\xi}_0 = \bar{s}(t) - 2t\gamma_1$ and (A3) has been used. The above relation together with (A6*a*) implies

$$|H_1| \leqslant \frac{A\,\mathrm{e}^{b\gamma_1}}{\sqrt{\pi t \delta_1}} |s(t) - \bar{s}(t)| < \frac{A\,\mathrm{e}^{b\gamma_1}}{\sqrt{\pi t \delta_1}} B_1 d\sqrt{\sigma} \equiv B_3 d\sqrt{\sigma}. \tag{A7}$$

We now consider (23*b*) together with (A2*c*); we get

$$|V_1| \leqslant \frac{M\lambda d}{\sqrt{\pi t \delta_1}} \left| \int_0^t \frac{1}{\delta_1} \exp\left[\frac{-(s(t)-s(t'))^2}{4\delta_1(t-t')}\right] \frac{\mathrm{d}t'}{\sqrt{t-t'}} \right| < \frac{MB_1 d}{\sqrt{\pi}\delta_1^{3/2}} \sqrt{\sigma} \equiv B_4 d\sqrt{\sigma}. \tag{A8}$$

Next, from (23*c*) we can write

$$|V_2| \leqslant \frac{1}{2\sqrt{\pi}} \frac{1}{\delta_1^{3/2}} M \int_0^t \left| \frac{(s(t) - \bar{s}(t)) - (s(t') - \bar{s}(t'))}{(t-t')} \right| \frac{\mathrm{d}t'}{\sqrt{t-t'}}$$
$$\leqslant \frac{1}{2\sqrt{\pi}} \frac{1}{\delta_1^{3/2}} \int_0^\sigma |\dot{s}(\vartheta) - \dot{\bar{s}}(\vartheta)| \frac{\mathrm{d}t'}{\sqrt{t-t'}} < \frac{MB_2 d\sqrt{\sigma}}{\sqrt{\pi}\delta_1^{3/2}} \equiv B_5 d\sqrt{\sigma} \tag{A9}$$

where the mean value theorem and (A6*b*) have been used.

Finaly, for the estimate of V_3 we put in (23*d*)

$$\tilde{Q} = -\frac{[(\bar{s}(t) - \bar{s}(t'))^2 - (s(t) - s(t'))^2]}{4\delta_1(t-t')}$$
$$= -\frac{1}{4\delta_1(t-t')} [(\bar{s}(t) - s(t)) - (\bar{s}(t') - s(t'))][(\bar{s}(t) - \bar{s}(t')) - (s(t) - s(t'))].$$

By using (A2*c*) and (A6*a*) we get

$$|\tilde{Q}| \leqslant \frac{B_1 M\lambda\,\mathrm{d}t|t-t'|}{\delta_1|t-t'|} < \frac{B_1 M\lambda}{\delta_1}\,\mathrm{d}\sigma \equiv B_6\,\mathrm{d}\sigma.$$

On the other hand, from (A2c) it also follows that

$$|\tilde{Q}| \leqslant \frac{2M^2\lambda^2}{\delta_1|t-t'|}|t-t'|^2 < \frac{2M^2\lambda^2}{\delta_1}\sigma < \frac{M\lambda}{\delta_1} \equiv B_7 \qquad (2M\lambda\sigma < 1).$$

From (23d) we then get

$$|V_3| \leqslant \frac{1}{2\sqrt{\pi}}\frac{1}{\delta_1^{3/2}}M\int_0^t \left|\frac{\bar{s}(t)-\bar{s}(t')}{(t-t')}\right| \frac{|1-\mathrm{e}^{-\tilde{Q}}|}{\sqrt{t-t'}}\,\mathrm{d}t'$$

$$\leqslant \frac{2M_2\lambda}{\sqrt{\pi}\delta_1^{3/2}}|\tilde{Q}|\,\mathrm{e}^{|\tilde{Q}|}\sqrt{\sigma} < \frac{MB_6\,\mathrm{e}^{B_7}}{\sqrt{\pi}\delta_1^{3/2}}\,\mathrm{d}\sqrt{\sigma}$$

which gives (24c)

$$|V_3| < B_8 d\sqrt{\sigma} \qquad \left(B_8 = \frac{MB_6\,\mathrm{e}^{B_7}}{\sqrt{\pi}\delta_1^{3/2}}\right).$$

Appendix B

In the system (42a)–(42b) the coefficients A_{ij} have the form

$$A_{11} = \frac{\delta_1 K_1}{\alpha_1\alpha_2} - \frac{V}{2\alpha_1\alpha_2}$$

$$A_{12} = \frac{\delta_2 K_2}{\beta_1\beta_2} + \frac{V}{2\beta_1\beta_2}$$

$$A_{21} = -\frac{3}{2}\frac{V}{\alpha_1\alpha_2}q + \frac{\delta_1 K_1}{\alpha_1\alpha_2}q + \frac{V^2K_1}{\alpha_1\alpha_2} + \frac{\lambda_1 q}{\delta_1} - V\left(\frac{1}{\delta_1} + \frac{\delta_1\gamma_1}{\alpha_1\alpha_2} + \frac{V^2}{4\delta_1\alpha_1\alpha_2}\right)$$

$$A_{22} = -\frac{\lambda_2}{\delta_2}q$$

with K_1 and K_2 given by (41d) and (41e); moreover it is

$$H = -\frac{V\delta_1}{\alpha_1\alpha_2}f_1(0) + \frac{\delta_1}{\alpha_1\alpha_2}f_1'(0) + \lambda_1 f_1(0) - \lambda_2 f_2(0).$$

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